

## A minor-monotone graph parameter based on oriented matroids

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### Abstract

For an undirected graph  $G = (V, E)$  let  $\lambda'(G)$  be the largest  $d$  for which there exists an oriented matroid  $M$  on  $V$  of corank  $d$  such that for each nonzero vector  $(x^+, x^-)$  of  $M$ ,  $x^+$  is nonempty and induces a connected subgraph of  $G$ .

We show that  $\lambda'(G)$  is monotone under taking minors and clique sums. Moreover, we show that  $\lambda'(G) \leq 3$  if and only if  $G$  has no  $K_5$ - or  $V_8$ -minor; that is, if and only if  $G$  arises from planar graphs by taking clique sums and subgraphs.

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### 1. Introduction

In [5] the following invariant  $\lambda(G)$  for a graph  $G = (V, E)$  was introduced:  $\lambda(G)$  is equal to the largest dimension of any linear subspace  $X$  of  $\mathbb{R}^V$  with the property that for any nonzero  $x \in X$  the graph  $\langle \text{supp}_+(x) \rangle$  induced by  $\text{supp}_+(x)$  is nonempty and connected. (Here  $\text{supp}_+(x)$  denotes the *positive support* of  $x$ ; that is, the set  $\{v \in V \mid x(v) > 0\}$ . Similarly,  $\text{supp}_-(x)$  denotes the *negative support* of  $x$ ; that is, the set  $\{v \in V \mid x(v) < 0\}$ . Moreover, for any  $U \subseteq V$ ,  $\langle U \rangle$  denotes the subgraph of  $G$  induced by  $U$ ; that is, the subgraph with vertex set  $U$  and edges all edges of  $G$  contained in  $U$ . In this paper, all graphs are assumed to be simple.)

This graph parameter can be easily seen to be monotone under taking minors. That is, if  $G$  is a minor of  $H$ , then  $\lambda(G) \leq \lambda(H)$ . So for each natural number  $d$  the class of graphs  $G$  with  $\lambda(G) \leq d$  is closed under taking minors.

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In [5] it is also shown that  $\lambda(G) = \max\{\lambda(G_1), \lambda(G_2)\}$  if  $G$  is a *clique sum* of  $G_1$  and  $G_2$  (that is, arises by identifying two cliques of equal size in  $G_1$  and  $G_2$ ). It was shown that

- (i)  $\lambda(G) \leq 1$  if and only if  $G$  is a forest;
- (ii)  $\lambda(G) \leq 2$  if and only if  $G$  is series-parallel;
- (iii)  $\lambda(G) \leq 3$  if and only if  $G$  arises by taking subgraphs and clique sums from planar graphs. (1)

The function  $\lambda(G)$  was motivated by the graph invariant  $\mu(G)$  introduced by Colin de Verdière [2] (cf. [3]), although we do not know a relation between the two numbers. (It might be that  $\lambda(G) \leq \mu(G)$  holds for each graph  $G$ .)

In the discussion after presenting the results above at the *5ème Colloque International Graphes et Combinatoire* in Marseille Luminy (September 1995), the first author of the present paper raised the question of extending these results to oriented matroids. The present paper shows that indeed most results of [5] are maintained under such an extension.

We first give the definition of oriented matroid (see [1] for background). To this end it is convenient to introduce, for any ordered pair  $x = (a, b)$ , the notation  $x^+ := a$  and  $x^- := b$ .

Let  $M = (V, X)$  be an oriented matroid, where  $X$  is the set of ‘vectors’ of  $M$ . That is,  $X$  is a collection of ordered pairs  $x = (x^+, x^-)$  of subsets of  $V$  satisfying

- (i) for each  $x \in X$ ,  $x^+ \cap x^- = \emptyset$ ;
- (ii)  $\mathbf{0} := (\emptyset, \emptyset) \in X$ ;
- (iii) if  $x \in X$  then  $-x := (x^-, x^+) \in X$ ;
- (iv) if  $x, y \in X$ , then  $x \cdot y := (x^+ \cup (y^+ \setminus x^-), x^- \cup (y^- \setminus x^+)) \in X$ ;
- (v) if  $x, y \in X$  and  $u \in x^+ \cap y^-$ , then there exists a  $z \in X$  such that  $u \notin z^+ \cup z^-$ ,  $(x^+ \setminus y^-) \cup (y^+ \setminus x^-) \subseteq z^+ \subseteq x^+ \cup y^+$ , and  $(x^- \setminus y^+) \cup (y^- \setminus x^+) \subseteq z^- \subseteq x^- \cup y^-$ . (2)

The elements of  $X$  are called the *vectors* of the oriented matroid ( $\mathbf{0}$  is the *zero*). Any linear subspace  $Y$  of  $\mathbb{R}^V$  gives an oriented matroid  $(V, X)$ , by taking  $X := \{(\text{supp}_+(x), \text{supp}_-(x)) \mid x \in Y\}$ .

For any oriented matroid  $M = (V, X)$ , the minimal nonempty subsets of  $\{x^+ \cup x^- \mid x \in X\}$  form the circuit collection of a matroid, again denoted by  $M$ . Thus matroid terminology applies to oriented matroids. We give the concepts we need below, expressed in terms of the circuits of  $M$ .

The *rank* of a subset  $U$  of  $V$  is the size of a largest subset  $U'$  of  $U$  not containing a circuit of  $M$ . The *rank*  $\text{rank}(M)$  of  $M$  is the rank of  $V$ .

A *cobase* is a base of the dual matroid  $M^*$ ; that is, it is an inclusionwise minimal subset intersecting each circuit of  $M$ . The *cospan*  $\text{cospan}(U)$  of a subset  $U$  of  $V$  is the set of elements  $v \in V$  such that there is no circuit containing  $v$  and not intersecting  $U$  (so  $U \subseteq \text{cospan}(U)$ ). The *corank*  $\text{corank}(U)$  of a subset  $U$  of  $V$  is the size of a minimal subset  $U'$  of  $U$  such that  $U \subseteq \text{cospan}(U')$ . A basic matroid theory

formula is

$$\text{corank}(U) = |U| + \text{rank}(V \setminus U) - \text{rank}(V). \tag{3}$$

The *corank*  $\text{corank}(M)$  of  $M$  is equal to  $\text{corank}(V)$ , which is equal to  $|V| - \text{rank}(V)$ .

Finally, we denote the deletion and contraction of  $U$  by  $M \setminus U$  and  $M/U$ , respectively. In terms of oriented matroids, if  $M = (V, X)$  is an oriented matroid and  $U \subseteq V$ , then  $M \setminus U$  is the oriented matroid  $(V \setminus U, X')$  with  $X' := \{x \in X \mid (x^+ \cup x^-) \cap U = \emptyset\}$ , and  $M/U$  is the oriented matroid  $(V \setminus U, X'')$  with  $X'' := \{(x^+ \setminus U, x^- \setminus U) \mid x \in X\}$ .

We next describe our graph parameter based on oriented matroids. Let  $G = (V, E)$  be an undirected graph. A *valid representation* for  $G$  is any oriented matroid  $M = (V, X)$  with the property that for each nonzero  $x \in X$ , the subgraph  $\langle x^+ \rangle$  of  $G$  induced by  $x^+$  is nonempty and connected. Let  $\lambda'(G)$  be the largest corank of any valid representation for  $G$ .

As each subspace of  $\mathbb{R}^V$  gives an oriented matroid, with corank equal to the dimension of the subspace, we have for each graph  $G$

$$\lambda(G) \leq \lambda'(G). \tag{4}$$

One of the consequences of this paper is that there are no graphs with  $\lambda(G) \leq 3$  and  $\lambda(G) < \lambda'(G)$ . In fact, we do not know any graph  $G$  with strict inequality in (4).

## 2. $\lambda'$ is minor-monotone

We now first show:

**Theorem 1.** *If  $G$  is a minor of  $H$  then  $\lambda'(G) \leq \lambda'(H)$ .*

**Proof.** Let  $M = (V, X)$  be a valid representation of  $G = (V, E)$  with  $\text{corank}(M) = \lambda'(G)$ . If  $G$  arises from  $H$  by deleting an edge of  $G$ , then  $M$  is also a valid representation for  $H$ . So  $\lambda'(H) \geq \text{corank}(M) = \lambda'(G)$ .

If  $G$  arises from  $H$  by contracting an edge  $e = uv$  of  $H$  to vertex  $w$  of  $G$ , then replacing in any  $x \in X$ ,  $x^+$  by  $(x^+ \setminus \{w\}) \cup \{u, v\}$  if  $w \in x^+$ , and similarly,  $x^-$  by  $(x^- \setminus \{w\}) \cup \{u, v\}$  if  $w \in x^-$ , gives a valid representation  $M'$  for  $H$ , with  $\text{corank}(M') = \text{corank}(M) = \lambda'(G)$ .  $\square$

This theorem implies, by Robertson and Seymour's theorem [4], that for each fixed  $n$  there is a finite class  $\mathcal{F}_n$  of graphs with the property that for any graph  $G$ :  $\lambda'(G) \geq n$  if and only if  $G$  has no minor in  $\mathcal{F}_n$ .

We note that for the complete graph  $K_n$  one has

**Theorem 2.**  $\lambda'(K_n) = n - 1$ .

**Proof.** Let  $M = (V, X)$  be a valid representation for  $K_n$ . If  $\text{corank}(M) = n$ , then  $\text{rank}(M) = 0$ , and therefore  $\{v\}$  is a circuit for each  $v \in V$ . So  $\{v\}$  contains  $x^+ \cup x^-$

for some nonzero  $x \in X$ . This contradicts the fact that both  $x^+$  and  $x^-$  are non-empty.

On the other hand, the set  $X$  of all pairs  $(U, W)$  with  $U = \emptyset = W$  or  $U \neq \emptyset \neq W$  and  $U \cap W = \emptyset$ , gives a valid representation for  $K_n$  of corank  $n - 1$ .  $\square$

So Hadwiger's conjecture implies the conjecture that  $\gamma(G) \leq \lambda'(G) + 1$  for each graph  $G$ , where  $\gamma(G)$  is the chromatic number of  $G$ . (Hadwiger's conjecture states that  $\gamma(G) \leq n$  if  $G$  does not have any  $K_{n+1}$ -minor.)

It is useful to note:

**Theorem 3.** *If graph  $G'$  arises from graph  $G$  by deleting one vertex, then  $\lambda'(G) \leq \lambda'(G') + 1$ .*

**Proof.** Let  $M = (V, X)$  be a valid representation for  $G = (V, E)$ , of corank  $\lambda'(G)$ . Let  $G'$  arise from  $G$  by deleting vertex  $v$ . Then the matroid  $M' := M \setminus \{v\}$  obtained from  $M$  by deleting  $v$  is a valid representation for  $G'$ . Moreover  $\text{rank}(M') \leq \text{rank}(M)$ , and hence  $\text{corank}(M') = |V| - 1 - \text{rank}(M') \geq |V| - 1 - \text{rank}(M) = \lambda'(G) - 1$ .  $\square$

### 3. Clique sums

In this section we show that the function  $\lambda'(G)$  does not increase by taking clique sums, and from this we directly derive characterizations of the classes of graphs  $G$  satisfying  $\lambda'(G) \leq 1$  and  $\lambda'(G) \leq 2$ . We first prove a lemma on oriented matroids.

**Lemma 1.** *Let  $M = (V, X)$  be an oriented matroid and let  $x, y \in X$  with  $\emptyset \neq x^+ \subseteq y^-$  and  $x \neq -y$ . Then there is a nonzero  $z \in X$  such that  $z^+ \subseteq y^+$  and  $x^+ \not\subseteq z^-$ .*

**Proof.** Choose a nonzero  $z \in X$  such that (i)  $x^+ \not\subseteq z^-$ , (ii)  $z^+ \subseteq x^+ \cup y^+$ , (iii)  $x^- \setminus y^+ \subseteq z^- \subseteq x^- \cup y^-$ , and (iv)  $|y^+ \cup z^+|$  as small as possible. Such a  $z$  exists, since  $z = x$  satisfies (i)–(iii).

Assume  $z^+ \not\subseteq y^+$ , and choose  $u \in z^+ \setminus y^+$ . So  $u \in x^+$ , and hence  $u \in y^-$ . Therefore, applying (2)(v) to  $y, z$ , there is a  $z' \in X$  such that  $u \notin z'^+ \cup z'^-$ ,  $z'^+ \subseteq y^+ \cup z^+$ ,  $z'^- \subseteq y^- \cup z^-$ ,  $(y^+ \setminus z^-) \cup (z^+ \setminus y^-) \subseteq z'^+$ , and  $(z^- \setminus y^+) \cup (y^- \setminus z^+) \subseteq z'^-$ . Then  $x^+ \not\subseteq z'^-$  (as  $u \notin z'^-$ ),  $z'^+ \subseteq y^+ \cup z^+ \subseteq x^+ \cup y^+$ ,  $x^- \setminus y^+ \subseteq z^- \setminus y^+ \subseteq z'^- \subseteq y^- \cup z^- \subseteq x^- \cup y^-$ , and  $y^+ \cup z'^+ \subseteq y^+ \cup z^+$  (as  $u \notin y^+ \cup z'^+$ ). Since  $|y^+ \cup z^+|$  is minimal it follows that  $z' = \mathbf{0}$ . Hence  $y^+ \subseteq z^-$ ,  $z^+ \subseteq y^-$ ,  $z^- \subseteq y^+$ , and  $y^- \subseteq z^+$ . So  $z = -y$ , and therefore  $y^- \subseteq x^+$  and  $y^+ \subseteq x^-$ . Moreover,  $x^- \setminus y^+ \subseteq y^+$ , and hence  $x^- \subseteq y^+$ . So  $x = -y$ , contradicting our assumption.  $\square$

The lemma is used to prove

**Theorem 4.** *Let  $M = (V, X)$  be a valid representation for  $G = (V, E)$  and let  $y, z \in X$ . If  $y \neq -z$  then  $\langle y^+ \cup z^+ \rangle$  is connected.*

**Proof.** Suppose  $y \neq -z$  and  $\langle y^+ \cup z^+ \rangle$  is disconnected. So  $y$  and  $z$  are nonzero and  $y^+ \cap z^+ = \emptyset$ . Consider  $z \cdot y = (z^+ \cup (y^+ \setminus z^-), z^- \cup (y^- \setminus z^+))$ . Since  $\langle z^+ \cup (y^+ \setminus z^-) \rangle$  is connected,  $y^+ \setminus z^- = \emptyset$ , that is,  $y^+ \subseteq z^-$ . This implies by Lemma 1 that there is a nonzero  $w \in X$  such that  $w^+ \subseteq z^+$  and  $y^+ \not\subseteq w^-$ . Consider  $w \cdot y = (w^+ \cup (y^+ \setminus w^-), w^- \cup (y^- \setminus w^+))$ . Then  $w^+$  is a nonempty subset of  $z^+$  and  $y^+ \setminus w^-$  is a nonempty subset of  $y^+$ , contradicting the fact that  $\langle w^+ \cup (y^+ \setminus w^-) \rangle$  is connected.  $\square$

This theorem does not apply if  $y = -z$ . This case can be described as follows.

**Theorem 5.** *Let  $M = (V, X)$  be a valid representation for  $G = (V, E)$ . Then for all  $y \in X$  with  $\langle y^+ \cup y^- \rangle$  not connected, there exist  $\text{corank}(M)$  pairwise openly vertex-disjoint paths connecting  $y^+$  and  $y^-$ , except if  $\text{corank}(M) = 1$  and  $y^+$  and  $y^-$  are contained in different components of  $G$ .*

**Proof.** Suppose not. Then by Menger’s theorem there exists a subset  $U$  of  $V$  such that  $y^+$  and  $y^-$  are contained in different components of  $G - U$  and such that  $|U| < \text{corank}(M)$ . By Theorem 4,  $y^+ \cup y^-$  is the unique circuit of  $M$  contained in  $V \setminus U$ . (Indeed, by Theorem 4, for any nonzero  $x \in X$  and  $x^+ \cup x^- \subseteq V \setminus U$  one has  $x \in \{y, -y\}$ .) Therefore  $\text{rank}(V \setminus U) = |V \setminus U| - 1$ .

If  $U = \emptyset$  then  $\text{rank}(M) = |V| - 1$ , and hence  $\text{corank}(M) = 1$ . If  $U \neq \emptyset$ , we can choose some  $u \in U$ . Let  $x \in X$  be such that  $x^+ \cup x^- \subseteq (V \setminus U) \cup \{u\}$ . If  $u \notin x^+$  then  $x \in \{y, -y\}$  (by Theorem 4). Similarly, if  $u \notin x^-$  then again  $x \in \{y, -y\}$ . Concluding,  $y^+ \cup y^-$  is the unique circuit contained in  $(V \setminus U) \cup \{u\}$  and hence  $\text{rank}((V \setminus U) \cup \{u\}) = |V \setminus U|$ . Hence  $\text{rank}(M) \geq \text{rank}((V \setminus U) \cup \{u\}) = |V \setminus U|$ . This contradicts the fact that  $\text{rank}(M) = |V| - \text{corank}(M) = |V| - d < |V \setminus U|$ .  $\square$

We use Theorems 4 and 5 to investigate the behaviour of  $\lambda'(G)$  upon taking a ‘clique sum’, which is defined as follows. Let  $G = (V, E)$  be a graph and let  $V_1$  and  $V_2$  be subsets of  $V$  such that  $V = V_1 \cup V_2$ ,  $K := V_1 \cap V_2$  is a clique in  $G$  and such that there is no edge connecting  $V_1 \setminus K$  and  $V_2 \setminus K$ . Then  $G$  is called a *clique sum* of  $G_1 := \langle V_1 \rangle$  and  $G_2 := \langle V_2 \rangle$ .

**Theorem 6.** *If  $G$  is a clique sum of  $G_1$  and  $G_2$  then  $\lambda'(G) = \max\{\lambda'(G_1), \lambda'(G_2)\}$ , except if  $G = \bar{K}_2$ .*

**Proof.** Since  $G_1$  and  $G_2$  are subgraphs of  $G$ , we have  $\lambda'(G) \geq \max\{\lambda'(G_1), \lambda'(G_2)\}$ . So it suffices to show that  $\lambda'(G) = \lambda'(G_i)$  for  $i = 1$  or  $2$ . Assume that  $\lambda'(G) > \max\{\lambda'(G_1), \lambda'(G_2)\}$ . Let  $d := \lambda'(G)$ ,  $G = (V, E)$ ,  $G_1 = (V_1, E_1)$ ,  $G_2 = (V_2, E_2)$ ,  $K := V_1 \cap V_2$ , and  $t := |K|$ . We may assume that we have chosen this counterexample so that  $t$  is as small as possible.

Then  $\langle V_1 \setminus K \rangle$  has a component  $L$  such that each vertex in  $K$  is adjacent to at least one vertex in  $L$ . Otherwise  $G$  would be a repeated clique sum of subgraphs of  $G_1$  and  $G_2$  with common clique sum smaller than  $t$ . In that case  $\lambda'(G) = \max\{\lambda'(G_1), \lambda'(G_2)\}$

would follow by the minimality of  $t$ . Concluding,  $G_1$  has a  $K_{t+1}$ -minor, and therefore  $\lambda'(G_1) \geq t$ . Hence  $\lambda'(G) > t$ .

Let  $M = (V, X)$  be a valid representation for  $G$  with  $\text{corank}(M) = d$ . There exists a nonzero  $y \in X$  such that  $y^+ \cup y^- \subseteq V \setminus K$  (otherwise  $\text{rank}(M) \geq |V \setminus K|$ , contradicting the fact that  $d > t$ ).

By Theorem 5 both  $y^+$  and  $y^-$  are contained in the same component of  $G - K$ . Hence we may assume that  $y^+ \cup y^- \subseteq V_1 \setminus K$ . Hence by Theorem 4 we have that there is no nonzero  $x \in X$  with  $x^+ \subseteq V_1 \setminus K$ . So  $M/(V_1 \setminus K)$  has corank equal to  $\text{corank}(M)$ . Moreover, for each nonzero  $x \in X$ ,  $x^+ \cap V_2$  induces a nonempty connected subgraph of  $G_2$ . Hence  $\lambda'(G_2) \geq \text{corank}(M) = \lambda'(G)$ , contradicting our assumption that  $\lambda'(G_2) < d$ .  $\square$

This theorem directly implies characterizations of those graphs  $G$  satisfying  $\lambda'(G) \leq 1$  and  $\lambda'(G) \leq 2$ .

**Corollary 6a.** *For any graph  $G$ ,  $\lambda'(G) \leq 1$  if and only if  $G$  does not have a  $K_3$ -minor; that is, if and only if  $G$  is a forest.*

**Proof.** If  $\lambda'(G) \leq 1$  then  $G$  has no  $K_3$ -minor, as  $\lambda'(K_3) = 2$ . Conversely, if  $G$  is a forest, then  $G$  arises by taking clique sums and subgraphs from the graph  $K_2$ . As  $\lambda'(K_2) = 1$ , Theorem 6 gives the corollary.  $\square$

**Corollary 6b.** *For any graph  $G$ ,  $\lambda'(G) \leq 2$  if and only if  $G$  does not have a  $K_4$ -minor; that is, if and only if  $G$  is a series-parallel graph.*

**Proof.** If  $\lambda'(G) \leq 2$  then  $G$  has no  $K_4$ -minor, as  $\lambda'(K_4) = 3$ . Conversely, if  $G$  is a series-parallel graph, then  $G$  arises by taking clique sums and subgraphs from the graph  $K_3$ . As  $\lambda'(K_3) = 2$ , Theorem 6 gives the corollary.  $\square$

#### 4. Graphs satisfying $\lambda'(G) \leq 3$

We characterize in this section the graphs  $G$  satisfying  $\lambda'(G) \leq 3$ . The main step consists in proving that  $\lambda'(G) \leq 3$  if  $G$  is planar.

**Theorem 7.** *If  $G$  is planar then  $\lambda'(G) \leq 3$ .*

**Proof.** Suppose  $G = (V, E)$  is a planar graph with  $\lambda'(G) \geq 4$  and  $|V|$  minimal. We assume that we have an embedding of  $G$  in the sphere. For each face  $f$  of  $G$  let  $V_f$  be the set of vertices incident with  $f$ . Note that  $G$  is 4-connected, since otherwise it would be a subgraph of clique sums of smaller planar graphs, and hence we would have  $\lambda'(G) \leq 3$  by Theorem 6.

Let  $M = (V, X)$  be a valid representation for  $G$  with  $\text{corank}(M) \geq 4$ . Then  $\text{corank}(\{u\}) = 1$  for each  $u \in V$ ; that is,  $u$  is contained in at least one circuit of  $M$ . Otherwise, we can delete  $u$  from  $G$  and  $M$ .

We may assume that, for each edge  $uv$ ,  $\text{corank}(\{u, v\}) = 2$ ; that is, there is a circuit containing  $u$  but not  $v$ . Otherwise, either for each  $x \in X$  one has  $u \in x^+ \Leftrightarrow v \in x^-$ , in which case we can delete the edge  $\{u, v\}$  from  $G$ , or for each  $x \in X$  one has  $u \in x^+ \Leftrightarrow v \in x^+$ , in which case we can contract the edge  $\{u, v\}$  in  $G$  and identify elements  $u$  and  $v$  in  $M$ .

Note that this implies that if  $f$  and  $f'$  are adjacent faces (that is, have an edge in common) and  $\text{corank}(V_f) = 2 = \text{corank}(V_{f'})$ , then  $\text{cospan}(V_f) = \text{cospan}(V_{f'})$ .

Fixing  $V$  we choose  $E$  maximal under the condition that  $\text{corank}(\{u, v\}) = 2$  for each edge  $\{u, v\}$ . Then  $\text{corank}(V_f) \in \{2, 3\}$  for each face  $f$ . Indeed,  $\text{corank}(V_f) \geq 2$ , as each edge  $e$  has  $\text{corank}(e) \geq 2$ . Moreover, if  $\text{corank}(V_f) \geq 4$ ,  $V_f$  contains at least two nonadjacent vertices  $u, v$  with  $\text{corank}(\{u, v\}) = 2$ . This contradicts the maximality of  $E$ .

For  $x \in X$  let  $\mathcal{F}_x$  be the set of faces  $f$  for which  $V_f \cap x^+ \neq \emptyset$  and  $V_f \cap x^- \neq \emptyset$ . Then:

Let  $f$  and  $f'$  be two faces with  $\text{corank}(V_f \cup V_{f'}) \geq 4$ .

Then there is an  $x \in X$  with  $f, f' \in \mathcal{F}_x$ . (5)

As  $\text{corank}(V_f) \geq 2$ ,  $\text{corank}(V_{f'}) \geq 2$ , and  $\text{corank}(V_f \cup V_{f'}) \geq 4$ , there exist  $u, v \in V_f$ ,  $u', v' \in V_{f'}$  with  $\text{corank}(\{u, v, u', v'\}) = 4$ . Therefore, we can find  $x \in X$  such that  $u, u' \in x^+$  and  $v, v' \in x^-$ . So  $f, f' \in \mathcal{F}_x$ , proving (5).

For  $x \in X$  let  $W_x := \bigcup \{V_f \mid f \in \mathcal{F}_x\}$ . We show:

$\text{corank}(W_x) \leq 3$  for all  $x \in X$ . (6)

Note that (6) implies an immediate contradiction with (5), as  $\text{corank}(V) \geq 4$ .

We show that (6) holds. It suffices to show the result for  $x \in X$  such that  $x^+ \cup x^- = V$ . (Indeed, if there exists  $u \notin x^+ \cup x^-$ , let  $y \in X$  with  $u \in y^+$  and set  $z := x \cdot y$ . Then,  $z^+ \supseteq x^+ \cup \{u\}$ ,  $z^- \supseteq x^-$  and  $W_z \supseteq W_x$ . Hence validity of the result for  $z$  will imply validity for  $x$ .)

Let  $x \in X$  with  $x^+ \cup x^- = V$  be given. Observe that if  $f$  and  $f'$  are faces with  $\text{corank}(V_f) = \text{corank}(V_{f'}) = 2$  and having a common edge,  $e$  say, then  $\text{cospan}(V_f) = \text{cospan}(V_{f'})$ , as it is equal to  $\text{cospan}(e)$ . Similarly,  $\text{cospan}(V_f) \subseteq \text{cospan}(V_{f'})$  if  $\text{corank}(V_f) = 2$ ,  $\text{corank}(V_{f'}) = 3$  and  $f, f'$  share a common edge.

As both  $\langle x^+ \rangle$  and  $\langle x^- \rangle$  are connected, the cut  $\delta(x^+)$  corresponds in the dual graph of  $G$  to a circuit  $C$  which traverses exactly two edges in each face  $f \in \mathcal{F}_x$ .

Suppose, to obtain a contradiction, that  $\text{corank}(W_x) \geq 4$ . Then there exist faces  $f, f' \in \mathcal{F}_x$  with  $\text{corank}(V_f) = \text{corank}(V_{f'}) = 3$  and such that  $\text{cospan}(V_f) \neq \text{cospan}(V_{f'})$ . They correspond to two nodes on  $C$ . Denote by  $f_1, \dots, f_t$  the faces between  $f$  and  $f'$  when traveling from  $f$  to  $f'$  along  $C$  (in a given direction). Then we may assume that  $\text{corank}(V_{f_i}) = 2$  for all  $i = 1, \dots, t$ . For  $i = 0, 1, \dots, t$ , let  $u_i v_i$  be the edge common to the faces  $f_i$  and  $f_{i+1}$ , setting  $f_0 := f$  and  $f_{t+1} := f'$ . So each  $u_i v_i$  belongs to  $\delta(x^+)$  (as  $G$  is 4-connected). We may assume that  $u_i \in x^+$  and  $v_i \in x^-$  for each  $i$ .

Now choose  $w \in V_f \setminus \text{cospan}(V_{f'})$  and  $w' \in V_{f'} \setminus \text{cospan}(V_f)$ . Then the set  $\{u_0, v_0, w, w'\}$  has corank 4. Hence, there exists  $y \in X$  such that  $w, w' \in y^+$  and  $u, v \notin y^+ \cup y^-$ . Hence, the set  $y^+ \cup y^-$  contains none of the vertices on the faces  $f_1, \dots, f_t$  (since  $V_{f_i} \subseteq \text{cospan}(\{u_0, v_0\})$  for all  $i = 1, \dots, t$ ). In particular,  $u_i, v_i \notin y^+ \cup y^-$  for  $i = 1, \dots, t$ . By connectivity of  $\langle y^+ \rangle$  there exists a path  $P$  from  $w$  to  $w'$  which is entirely contained in  $y^+$ .

Consider the region  $R := \bigcup_{i=0}^{t+1} f_i$  (where faces are assumed to be topologically closed). As  $P$  joins two nodes on the boundary of  $R$ ,  $R \cup P$  partitions the rest of the sphere into two regions  $R_1$  and  $R_2$ . We choose indices such that  $R_1$  has the vertices  $u_0, \dots, u_t$  on its boundary, while  $R_2$  has the vertices  $v_0, \dots, v_t$  on its boundary.

By the connectivity of  $\langle y^- \rangle$ ,  $y^-$  is contained either in  $\bar{R}_1$  or in  $\bar{R}_2$ . Suppose first that  $y^-$  is contained in  $\bar{R}_1$ . Consider the element  $z := y \cdot x$  of  $X$ . Then,  $z^- \supseteq \{v_0, \dots, v_t\} \cup y^-$ , while  $u_0, \dots, u_t \in z^+$ . Then there is no path joining  $v_0$  and  $y^-$  which is entirely contained in  $z^-$ , contradicting the connectivity of  $\langle z^- \rangle$ .

Suppose next that  $y^-$  is contained in  $\bar{R}_2$ . Set  $z := y \cdot (-x)$ . Then we arrive similarly at a contradiction.  $\square$

We can now characterize the graphs  $G$  satisfying  $\lambda'(G) \leq 3$ . It follows from Theorems 6 and 7 that  $\lambda'(G) \leq 3$  if  $G$  can be obtained from planar graphs by taking clique sums and subgraphs. On the other hand, it follows from a result by Wagner [6] that the graphs that can be obtained from planar graphs by taking clique sums and subgraphs are precisely the graphs with no  $K_5$ - or  $V_8$ -minor. ( $V_8$  is the graph with vertices  $v_1, \dots, v_8$ , where  $v_i$  and  $v_j$  are adjacent if and only if  $|i - j| \in \{1, 4, 7\}$ .) It is shown in [5] that  $\lambda(V_8) = 4$ . Hence  $\lambda'(V_8) \geq 4$ . As deleting any vertex of  $V_8$  gives a planar graph, Theorem 3 implies that  $\lambda'(V_8) = 4$ . Moreover, by Theorem 2  $\lambda'(K_5) = 4$ . Therefore,

**Theorem 8.** *A graph  $G$  satisfies  $\lambda'(G) \leq 3$  if and only if  $G$  has no  $K_5$ - or  $V_8$ -minor; that is, if and only if  $G$  can be obtained from planar graphs by taking clique sums and subgraphs.*

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